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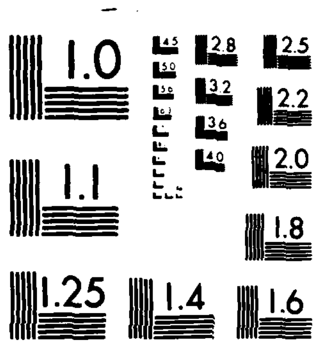

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Life Distribution Properties of Devices

Subject to a Lévy Wear Process

by

M. Abdel-Hameed
Department of Mathematics
University of North Carolina at Charlotte

Technical Report No. 8

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ABSTRACT

Assume that a device is subject to wear. Over time the wear is assumed to be an increasing Lévy process (X_t). Suppose the device has a threshold Y with right-tail probability $\overline{G}$. Let ζ be the failure time of the device and $\overline{F}_x$ be its survival probability given that $X_0 = x$. It is shown that life distribution properties of $\overline{G}$ are inherited as corresponding properties of $\overline{F}_x$ for each $x \in R_+$. Optimal replacement policies for such devices are discussed for suitably chosen cost functions when the failure rate of $\overline{G}$ is bounded and continuous a.e. on the support of $\overline{G}$.

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ABSTRACT

Assume that a device is subject to wear. Over time the wear is assumed to be an increasing Lévy process (X_t). Suppose the device has a threshold Y with right-tail probability $\overline{G}$. Let ζ be the failure time of the device and $\overline{F}_x$ be its survival probability given that $X_0 = x$. It is shown that life distribution properties of $\overline{G}$ are inherited as corresponding properties of $\overline{F}_x$ for each $x \in R_+$. Optimal replacement policies for such devices are discussed for suitably chosen cost functions when the failure rate of $\overline{G}$ is bounded and continuous a.e. on the support of $\overline{G}$.

A

1. INTRODUCTION.

Let (Ω, F, P) be a probability space. On it, $X = (X_t)$ is an increasing, right continuous, pure jump process with independent incredments, $X_0 \equiv 0$. Every such X has the form (see ITO(1969))

$$X_t = \sum_{s \le t} (X_s - X_{s-}) ,$$

and for every Borel set $B \subset [0,\infty) \times (0,\infty)$,

$$N(B) = \sum_{s \ge 0} 1_B(s, X_s - X_{s-})$$

has the Poisson distribution with some mean $n(B)$; it follows then that $N(B_1), \dots, N(B_n)$ are independent whenever $B_1, \dots, B_n$ are disjoint. We assume that the mean measure n has the form

$$n(ds,dy) = \Lambda(ds)\mu(dy) , \quad s > 0, \; y > 0.$$

Then, μ is necessarily a Lévy measure, that is, μ satisfies

$$\int \mu(dy)\,(y \wedge 1) < \infty \quad ,$$

which in turn implies that $\mu((\epsilon,\infty)) < \infty$ for every $\epsilon > 0$, and Λ is an arbitrary measure on $(0,\infty)$ with $0 \le \Lambda[0,t] < \infty$ for every t . Finally, we assume that the function $t \to \Lambda(t) \equiv \Lambda[0,t]$ is continuous (necessarily increasing and has $\Lambda(0) = 0$).

Since n is assumed to have the product form $\Lambda \times \mu$, it is possible to do a deterministic time change to cause X to have stationary and independent increments. Let τ be the right continuous functional inverse of Λ , i.e.,

$$\tau(u) = \inf\{t:\Lambda(t) > u\} , \quad u\epsilon R_+ ,$$

and set

$$\hat{X}_u = X_{\tau(u)} \qquad , \qquad u\epsilon R_+ .$$

Then, $\hat{X}$ is an increasing pure jump process with stationary and independent increments, a Lévy process. If $\hat{N}$ is defined for $\hat{X}$ as N was for X , then $\hat{N}$ is a Poisson random measure with mean measure $\hat{n}(ds,dy) = ds\mu(dy)$.

For x in R_+ the probability measure P^x is determined by

$$P^x\{X_{t_i}\epsilon A_i, i=1,\ldots,n\} = P\{X_{t_i}\epsilon A_i-x, i=1,\ldots,n\}$$

and

$$P^x\{X_0=x\} = 1.$$

Let (Ω',F',P') be another probability space and Y be a nonnegative random variable defined on it. For each x in R_+ define

$$P'^x = P'(\ |Y>x) .$$

The measure $\tilde{P}^x$ on $\tilde{\Omega} = \Omega \times \Omega'$ is defined as the product measure $\tilde{P}^x = P^x \times P'^x$, $E^x(\tilde{E}^x)$ denotes the expectation w.r.t. $P^x(\tilde{P}^x)$ and $E \equiv E^0$.

We interpret X_t as the amount of deterioration, or wear, suffered by the device during $[0,t]$, and Y as the threshold strength of the device, so that

$$\zeta = \inf\{t: X_u \geq Y\}$$

is the failure time of the device. Define, $\bar{G}(x) = P'(Y>x)$ and $A = \{x: \bar{G}(x) \neq 0\}$. Note that if

$$\hat{\zeta} = \inf\{u: \hat{X}_u \geq Y\} ,$$

then

$$\hat{\zeta} = \Lambda(\zeta) \quad \text{and} \quad \zeta = \tau(\hat{\zeta}) \quad \text{almost surely. It follows that}$$

$$\bar{F}_x(t) = \tilde{P}^x\{\zeta > t\} = \tilde{P}^x\{Y > X_t\}$$

$$(1.1) \qquad = \begin{cases} E^x[\bar{G}(X_t)]/\bar{G}(x) & , x \in A \\ 0 & , x \notin A . \end{cases}$$

Moreover,

$$\bar{F}_x(t) = \hat{\bar{F}}_x(\Lambda(t)) \quad ,$$

where

$$\hat{\bar{F}}_x(u) = \tilde{P}^x\{\hat{\zeta} > u\}$$

$$(1.2) \qquad = \begin{cases} E^x[\bar{G}(\hat{X}_u)]/\bar{G}(x) & , x \in A \\ 0 & , x \notin A , \end{cases}$$

is the survival probability corresponding to $\hat{\zeta}$.

In Section II we show that life distribution properties of the threshold Y are inherited as corresponding properties of the failure time ζ, under appropriate conditions on the function $\Lambda(t)$.

In Section III we determine the form of the optimal replacement policies for such devices for properly chosen cost functions.

Throughout, the word "increasing" will be used to mean "nondecreasing" and "decreasing" will mean "nonincreasing".

II. PRESERVATION OF LIFE DISTRIBUTIONS.

Assume that a device is subject to wear occuring according to the increasing, right continuous, pure jump process with independent increments described in Section I. The device has a certain threshold Y defined on (Ω', F', P'). The device fails once the wear exceeds or equals to the threshold. Let $\bar{G}$ be the right tail probability of the threshold Y and $A = \{x: \bar{G}(x) \neq 0\}$. Then for any x in R_+ the survival probability of the device is defined by equation (1.1).

In this section we prove that life distribution properties of the threshold right tail probability $\bar{G}$ are inherited as corresponding properties of the survival probability $\bar{F}_x$ under suitable conditions on the function Λ.

<u>2.1. DEFINITION.</u> Let $x \in R_+$ and ζ be the failure time with survival probability $\bar{F}_x$. Define $U_x = \{t: \bar{F}_x(t) > 0\}$. Then

$\overline{F}_x$ is said to have:

(i) increasing (decreasing) failure rate if $\tilde{P}^x(\zeta > t + s | \zeta > t)$ is nonincreasing (nondecreasing) in $t \in U_x$ for each $s \in R_+$:

(ii) increasing (decreasing) failure rate average if $\tilde{E}^x f(\zeta) \leq (\geq) [\tilde{E}^x f^\alpha(\zeta|\alpha)]^{1/\alpha}$ for each $0 < \alpha < 1$ and each nondecreasing function $f: R \to R_+$.

(iii) new better (worse) than used if $\tilde{E}^x[f(\zeta)] \geq (\leq) \tilde{E}^x[f(\zeta - s)|\zeta > s]$ for each nondecreasing function $f: R \to R_+$ and each $s \in R_+$.

A detailed investigation of these classes is given in Barlow and Proschan [2].

The following shows that life distribution properties of the function $\overline{G}$ inherited as corresponding properties of the survival probability $\overline{F}_x$ given by (1.1), under appropriate conditions on the function Λ.

2.2. THEOREM. Let $\overline{F}_x$ be given by (1.1). Assume the Lévy measure μ is finite. Then the following hold:

(i) If $\overline{G}$ has an increasing failure rate and Λ is convex and $\mu << \text{Leb}$ with a density f that is Pólya frequency function of order two (PF_2), then $\overline{F}_x(t)$ has increasing failure rate.

(ii) If $\overline{G}$ has a decreasing failure rate and Λ is concave, then $\overline{F}_x$ has a decreasing failure rate for each $x \in R_+$

(iii) If $\overline{G}$ has increasing failure rate average and Λ is starshaped, then $\overline{F}_x$ has increasing failure rate average for each $x \in R_+$.

(iv) If $\overline{G}$ has decreasing failure rate average and Λ is antistarshaped and $\mu << \text{Leb}$ with PF_2 density, then $\overline{F}(t)$ has decreasing failure rate average.

(v) If $\overline{G}$ is new better than used and Λ is superadditive, then for each $\alpha \in (0,1)$ and $x, t, s \in R_+$ we have $\overline{F}_x(t+s) \le \overline{F}_{\alpha x}(t)\, \overline{F}_{(1-\alpha)x}(s)$. In particular $\overline{F}_0$ is new better than used.

(vi) If $\overline{G}$ is new worse than used and Λ is subadditive, then for each $x \in R_+$, $\overline{F}_x$ is new worse than used.

2.3 REMARK. By an argument similar to the one used in the proof of Theorem (3.1) of Abdel-Hameed and Proschan [1], it suffices to prove the properties of $\hat{F}$ in terms of those of $\overline{G}$ and μ, and then draw the conclusion for the non-stationary case by using the relation $\overline{F} = \hat{\overline{F}} \circ \Lambda$.

PROOF OF THEOREM 2.2 (i). Let $N(t)$ be the number of jumps of $\hat{X}$, during $(0,t]$ and $\mu^{(k)}$ is the k^{th} convolution of μ, then

$$\hat{\overline{F}}_x(t) = E^x\,[\mu^{(N(t))}\overline{G}(x)]/\overline{G}(x)$$

where for each $k \ge 0$, $\mu^{(k)}\overline{G}(x) = \int_{R_0} \overline{G}(x+y)\mu^{(k)}(dy)$, $R_0 = R_+\backslash\{0\}$.

Since $\hat{\overline{F}}_x(t) = E^x\, \overline{G}(\hat{X}_t)/\overline{G}(x)$, then it suffices to show that $\overline{K}_x(t) \equiv E^x\, \overline{G}(\hat{X}_t)$ has increasing failure rate. Since f is a Pólya frequency of order two, then it follows by an argument similar to the one used in proving Theorem 4.9 of Esary, Marshall and Proschan [6] that $f^k(x)$ is TP_2 in k and x. Since $\overline{G}$ has increasing failure rate implies that $\overline{G}_x(y) = \overline{G}(x+y)$ enjoys the same property for each x, then the result follows from Theorem 5.4 of Karlin [10], page 130.

PROOF OF (ii). It suffices to show that $\overline{K}_x$ has decreasing failure rate. It is evident that $\overline{K}_x$ is differentiable and has decreasing failure rate if and only if the function $f: [0,\infty) \to [0,1]$ given by $f_x(t) = \mu\, \overline{K}_x(t)/\overline{K}_x(t)$ is nondecreasing in $t \in R_+$ for each $x \in R_+$.

This is true if and only if for each $z \in R_0$, $x \in R_+$ such that $x + z \in A$,

$$D(t_1,t_2) = \overline{K}_{x+z}(t_1)\,\overline{K}_x(t_2) - \overline{K}_{x+z}(t_1)\,\overline{K}_x(t_2)$$

is non positive for $t_1 \le t_2$, t_1, $t_2 \in U_x$. By the "Basic Composition Formula" (Karlin [10], page 17) we have that

$$D(t_1,t_2) = \sum_{k_1 \le k_2}\sum \begin{vmatrix} P(t_1,k_1) & P(t_1,k_2) \\ P(t_2,k_1) & P(t_2,k_2) \end{vmatrix} \begin{vmatrix} \mu^{(k_1)}\overline{G}(x+z) & \mu^{(k_1)}\overline{G}(x) \\ \mu^{(k_2)}\overline{G}(x+z) & \mu^{(k_2)}\overline{G}(x) \end{vmatrix},$$

where $P(t,k)$ is the Poisson kernel which is totally positive of all orders. Therefore for each k_1, k_2 in the domain of the sum the first determinant is nonnegative and we only need to show that for such k_1 and k_2 the second determinant in the sum is nonpositive. This is true if for each $k \ge 0$, the function $\sigma(k,x) = \mu^{(k)}\overline{G}(x)$ has decreasing failure rate. But $\mu^{(k)}\overline{G}(x)$ is a mixture of $\overline{G}_y(x) \overset{def}{=} \overline{G}(x+y)$ with respect to $\mu^{(k)}$ and decreasing failure rate property is preserved under mixtures.

<u>PROOF OF (iii)</u>. Since $\overline{F}_x(t) = E^x\overline{G}(\hat{X}_t)/\overline{G}(x)$, and since $[\overline{G}(x)]^{-1/t}$ is decreasing in $t \ge 0$, then it suffices to show that $E^x\overline{G}(\hat{X}_t)$ has increasing failure rate average. It suffices to show that $\overline{K}_x(t)$ has increasing failure rate average. Since $\overline{K}_x(t) = E\,\overline{G}(\hat{X}_t + x)$ and the function $\overline{G}_x(y) = \overline{G}(x+y)$ has increasing failure rate average when $\overline{G}$ does, then the result is immediate from Theorem 5.2 of Esary, Marshall and Proschan [6].

<u>PROOF OF (iv)</u>. It suffices to show that $\overline{K}_0(t)$ has decreasing failure rate average. Since $\overline{G}$ has decreasing failure rate average if and only if $\overline{G}(x) - e^{-\lambda x}$ changes sign at most once and if once in the order +,-. Since the density f is PF_2, then the

compound Poisson kernel $\ell(t,x) = \sum_{k\geq 0} P(t,k)\, f^{(k)}(x)$ is totally positive of order two and therefore

$$\bar{K}_0(t) - e^{-\psi(\lambda)t} = E\,[\bar{G}(\hat{X}_t) - e^{-\lambda\hat{X}_t}]$$

changes sign at most once and if once in the order $-$, $+$, by "The Variation Diminishing Property", Karlin [10], page 21.

Note that the average of the failure rate is less than or equal to $\int_{R_0} u(dy) = \lim_{\lambda\to\infty} \psi(\lambda)$. The function ψ is increasing and $\psi(0) = 0$. Therefore, $\bar{K}_0$ has decreasing failure rate average.

<u>PROOF OF (v)</u>. It suffices to show that $\bar{K}_x(t)$ enjoys the desired property. Now write,

$$\begin{aligned}
\bar{K}_x(t+s) &= E^x[\bar{G}(\hat{X}_{t+s})] \\
&= E[\bar{G}(\hat{X}_{t+s} + x)] \\
&\qquad [\text{by definition of } \tilde{P}^x] \\
&= E[\bar{G}(\hat{X}_{t+s} - \hat{X}_s + \alpha x + \hat{X}_s + (1-\alpha)x)] \\
&\leq E[\bar{G}(\hat{X}_{t+s} - \hat{X}_s - \alpha x)\bar{G}(\hat{X}_s + (1-\alpha)x)] \\
&= E[\bar{G}(\hat{X}_t + \alpha x)]E[\bar{G}(\hat{X}_s + (1-\alpha)x)] \\
&\qquad [\text{by the stationarity and independence} \\
&\qquad \text{of the increments}] \\
&= E^{\alpha x}[\bar{G}(\hat{X}_t)]\, E^{(1-\alpha)x}[\bar{G}(\hat{X}_s)] \\
&= \bar{K}_{\alpha x}(t)\, \bar{K}_{(1-\alpha)x}(s)
\end{aligned}$$

<u>PROOF OF (vi)</u>. By an argument similar to the one used in proving (v) and reversing the direction of the inequalities, an operation justified since $\overline{G}$ is new worse than used instead of being new better than used we get for each $t, s \in R_+$ and $\alpha \in (0,1)$

$$\begin{aligned} \overline{K}_x(t+s) &\geq \overline{K}_{\alpha x}(t)\, \overline{K}_{(1-\alpha)x}(s) \\ &\geq \overline{K}_x(t)\, \overline{K}_x(s) \\ &\quad [\text{since } \overline{K}_x \text{ decreases in } x]\ . \end{aligned}$$

Therefore $\overline{K}_x$ is new worse than used for each $x \in R_+$.

The following Theorem is the key theorem in this section.

<u>2.4. THEOREM</u>. The results of Theorem 2.2 hold for infinite Lévy measures.

<u>PROOF</u>. We only prove (i), the other cases follow similarly. First write for $n \in N_+ \setminus \{0\}$

$$X_t^n = \int_{[0,t]\times(\frac{1}{n},\infty)} y\, N(ds,dy)$$

Observe that the Lévy measure associated with X^n is of the form, $\nu^n(dt,dy) = \Lambda(dt)\, \mu^n(dy)$ where $\mu^n(dy) = \mu(dy) I_{(\frac{1}{n},\infty)}(y)$.

Since μ is a Lévy measure, then $\mu(\frac{1}{n}, \infty) < \infty$ for each n and thus μ^n is a finite measure.

For n in $N_+ \setminus \{0\}$, let $\hat{X}^n_u = \hat{X}_{\tau(u)}$, $u \in R_+$.

By Theorem 3.2 it follows that $\bar{K}^n_x(t) \overset{\text{def}}{=} [E^x \bar{G}(\hat{X}^n_t)]$ has increarsing failure rate. We only need to show that $\bar{F}_x(t) = E^x[\bar{G}(X_t)]$ enjoys the same property. Now let, $D = \{x : \bar{G}(x)$ is discontinuous$\}$.

Note that $\hat{X}^n_t$ increases to $\hat{X}_t$ almost everywhere as n goes to infinity. Let $\bar{G}(x_-) = \lim_{y \uparrow x} \bar{G}(y)$, then $\bar{G}(\hat{X}^n_t)$ converges to $\bar{G}(\hat{X}_{t-})$ almost every where for each t in R_+. By the bounded convergence theorem we have that $E^x[\bar{G}(\hat{X}^n_t)]$ converges to $E^x[\bar{G}(\hat{X}_{t-})]$ for each $t > 0$. Since $\bar{G}$ has countable number of discontinuities then it follows from Theorem 1.C of Kesten [11] that $P^x(X_t \in D) = 0$ for each $t > 0$. Therefore, $E^x[\bar{G}(X_{t-})] = E^x[\bar{G}(X_t)] = \bar{K}_x(t)$. Since limit of increasing failure rate survival functions is an increasing failure rate survival function, the result follows.

The following theorem is a partial result, when $\bar{G}$ depends on x and t .

<u>3.5 THEOREM</u>. Let $\bar{F}_x$ be given by (1.1). Assume that $\bar{G}$ depends on x and t . Then the following holds:

(i) Suppose that the map $x \to \bar{G}(x,t)$ has increasing failure rate average for each $t \in R_+$, the map $t \to \bar{G}(x,t)$ is decreasing for each $x \in R_+$ and Λ is starshaped. Then $\bar{F}_x$ has increasing failure rate average for each $x \in R_+$;

(ii) Suppose that the map $x \to \bar{G}(x,t)$ has decreasing failure rate and the map $t \to \bar{G}(x,t)$ is increasing for each $x \in R_+$. Let Λ be antistarshaped and $\mu << \text{Leb}$ with PF_2 density. Then $\bar{F}_0$ has decreasing failure rate average.

(iii) Suppose that the map $(x,t) \to \bar{G}(x,t)$ satisfies the property $\bar{G}(x+y,\ t+s) \le \bar{G}(x,t)\ \bar{G}(y,s)$ for each $x, y, t, s \in R_+$. Let Λ be superadditive. Then, for each $x,t,s \in R_+$ and $\alpha \in (0,1)$, we have $\bar{F}_x(t+s) \le \bar{F}_{\alpha x}(t)\ \bar{F}_{(1-\alpha)x}(s)$. In particular $\bar{F}_0$ is new better than used.

(iv) Suppose that the map $(x,t) \to \bar{G}(x,t)$ satisfies the property $\bar{G}(x+y,\ t+s) \ge \bar{G}(x,t)\ \bar{G}(y,s)$ for each $x, y, t, s \in R_+$. Let Λ be subadditive. Then $\bar{F}_x$ is new worse than used.

PROOF OF (i). Define $\bar{H}_x(t) = E^x[\bar{G}(X_t,t)]$. It is sufficient to show that for each $\lambda \in R_+ : \bar{H}_x(t) - e^{-\lambda t}$ changes sign at most once and if once in the order $+$, $-$.
Fix λ and let t' be the point where such a change occurrs.
Define, $\bar{H}'_x(t) = E^x\ \bar{G}(X_t,t')$. By Theorems 2.2 and 2.4 it follows that $\bar{H}'_x(t) - e^{-\lambda t}$ changes sign once and in the order $+$, $-$. Thus, $\bar{H}'_x(t) \ge e^{-\lambda t}$ for $t \le t'$ and $\bar{H}'_x(t) \le e^{-\lambda t}$ for $t \ge t'$. Since $t \to \bar{G}(x,t)$ is decreasing in t for each $x \in R_+$, then $\bar{H}_x(t) \ge \bar{H}'_x(t)$ for $t \le t'$ while $\bar{H}_x(t) \le \bar{H}'_x(t)$ for $t \ge t'$.
Therefore, $\bar{H}_x(t) \ge e^{-\lambda t}$ for $t \le t'$ and $\bar{H}_x(t) \le e^{-\lambda t}$ for $t \ge t'$, i.e. $\bar{H}_x(t) - e^{-\lambda t}$ changes sign once and from $+$ to $-$.

PROOF OF (ii). It is necessary and sufficient to show that for each $\lambda \in R_+$, $\bar{F}_0(t) - e^{-\lambda t}$ changes sign at most once and if once in the order $-$, $+$. Fix λ and let t' be the point where such a change occurrs. Define, $\bar{K}'_0(t) = E\ \bar{G}(X_t,t')$. By virtue of Theorems 2.2 and 2.4 it follows that $\bar{K}'_0(t) - e^{-\lambda t}$ changes in the order $-$, $+$. Thus, $\bar{K}'_0(t) \le e^{-\lambda t}$ for $t \le t'$ and $\bar{K}'_0(t) \ge e^{-\lambda t}$ for $t \ge t'$. Since $t \to \bar{G}(x,t)$ is increasing in t for each $x \in R_+$, then $\bar{K}_0(t) \le \bar{K}'_0(t)$ for $t \le t'$ and $\bar{K}_0(t) \ge \bar{K}'_0(t)$ for $t \ge t'$.
Therefore $\bar{K}_0(t) \le e^{-\lambda t}$ for $t \le t'$ and $\bar{K}_0(t) \ge e^{-\lambda t}$ for $t \ge t'$.
Hence, $\bar{K}_0(t) - e^{-\lambda t}$ changes sign once in the order $-$, $+$.

PROOF OF (iii). We need to show that $\overline{H}_x(t+s) \le \overline{H}_{\alpha x}(t)\overline{H}_{(1-\alpha)x}(s)$.

Observe that $\overline{H}_x(t+s) = E^x[\overline{G}(\hat{X}_{\Lambda(t+s)}, t+s)]$

$$\le E[\overline{G}(\hat{X}_{\Lambda(t)+\Lambda(s)} + x,\ t+s)]$$

$$= E[\overline{G}(\hat{X}_{\Lambda(t)+\Lambda(s)} - \hat{X}_{\Lambda(s)} + \alpha x + \hat{X}_{\Lambda(s)} + (1-\alpha)x,\ t+s)]$$

$$\le E[\overline{G}(\hat{X}_{\Lambda(t)+\Lambda(s)} - \hat{X}_{\Lambda(s)} + \alpha x,\ t)\overline{G}(\hat{X}_{\Lambda(s)} + (1-\alpha)x, s)]$$

$$= E[\overline{G}(\hat{X}_{\Lambda(t)} + \alpha x, t)]\ E\ [\overline{G}(\hat{X}_{\Lambda(s)} + (1-\alpha)x, s)]$$

[By the stationarity and independence of the increments]

$$= E^{\alpha x}[\overline{G}(\hat{X}_{\Lambda(t)}, t]\ E^{(1-\alpha)x}[\overline{G}(\hat{X}_{\Lambda(s)}, s)]$$

$$= \overline{F}_{\alpha x}(t)\overline{F}_{(1-\alpha)x}(s)$$

PROOF OF (iv). The proof of (iv) is similar to the proof of (iii), and hence is omitted.

III. OPTIMAL REPLACEMENT POLICY.

In this section we deal with the problem of finding the optimal replacement policy that minimizes the average cost per unit time for devices subject to the type of wear processes described in Section I. Define

$$Z_t = \begin{cases} X_t, & t < \zeta \\ +\infty, & t \ge \zeta \end{cases}$$

and

$$\hat{Z}_t = \begin{cases} \hat{X}_t , & t < \hat{\zeta} \\ +\infty , & t \geq \hat{\zeta} \end{cases}$$

Note that

$$Z_t = \hat{Z}_{\Lambda(t)}$$

and

$$\hat{Z}_t = Z_{\tau(t)} , \quad \tau \in R_+ .$$

The process Z is obtained by killing the wear process X at the failure time of the device and sending it to "iternity". The process $\hat{Z}$ is obtained from $\hat{X}$ similarly. For $t \in R_+$ let $F_t = \sigma(Z_s, s \leq t)$ and $\hat{F}_t = \sigma(\hat{Z}_s, s \leq t)$.

Devices subject to the wear process Z are replaced at or after failure at a constant cost c. Replacements before failure depends on the accumulated wear at time of replacement and is denoted by $c(\cdot)$. Naturally we assume that the cost of replacement before failure does not exceed the cost of replacement at or after failure. We will deal only with Markovian replacement times with respect to the history (F_t). By a Markovian replacement time we simply mean a stopping time with respect to the cononical history (F_t).

For each Markovian replacement time τ the average cost per unit time ψ_τ is of the form

$$\psi_\tau(x) = \{\tilde{E}^x[c(Z_\tau)I(\tau<\zeta)] + c\tilde{P}^x(\tau \geq \zeta)\}/\tilde{E}^x(\tau) ,$$

For each x in R_+ let $c_1(x) = c - c(x)$, $x \in A$ and equals to zero for x in the complement of A with respect to $\bar{R}_+$. Note that

$$\psi_\tau(x) = \{-\tilde{E}^x[c(Z_\tau)] + c\}/\tilde{E}^x(\tau) .$$

We will give sufficient conditions on the cost function c that guarantees that the optimal replacement policy that minimizes the expected cost per unit time is a control policy. Since $\psi_\tau(x) \equiv 0$ for each τ for which $\tilde{E}^x(\tau) = +\infty$, we will only concern ourselves with those Markovian replacement times for which $\tilde{E}^x(\tau) < \infty$, $x \in R_+$. Denote the class of such replacement times by U.

3.1 REMARK. Let $\hat{U}$ be the class of Markovian stopping times, $\hat{\tau}$, with respect to the history $(\hat{F}_t)$ such that $E^x(\hat{\tau}) < \infty$, $x \in R_+$. Note that τ belongs to U if and only if $\Lambda(\tau)$ belongs to $\hat{U}$.

The following proposition gives sufficient condition for the finiteness of the expected failure time.

3.2. PROPOSITION. Assume that the map $x \to \overline{G}(x,t)$ has increasing failure rate average for each $t \in R_+$, and the map $t \to \overline{G}(x,t)$ is decreasing for each $x \in R_+$ and Λ is starshaped. Then $\tilde{E}^x(\zeta)$ is finite for all $x \in R_+$.

PROOF. From Theorem 2.5 we know that $\overline{F}_x(t) = \tilde{P}^x(\zeta > t)$ has increasing failure rate average. Therefore there exists λ and $t^o \in R_+$ such that $\overline{F}_x(t) \geq e^{-\lambda t}$ for $t < t^o$ and $\overline{F}_x(t) \leq e^{-\lambda t}$ for $t \geq t^o$. Therefore,

$$\begin{aligned}\tilde{E}^x(\zeta) &= \int_0^\infty \overline{F}_x(t)dt \\ &\leq \int_0^{t^o} \overline{F}_x(t)dt + \int_{t^o}^\infty e^{-\lambda t}dt \\ &< \infty .\end{aligned}$$

Let $b\mathcal{B}(A)$ be the class of bounded Borel measurable functions whose domain is A. Every function in this class is extended to $\overline{R}_+$ by defining it to be equal to zero outside A. We denote the class of such functions by $\mathcal{B}_0$. For the stationary process $\hat{Z}$ we define the semigroup Q_t on $\mathcal{B}_0$ by $Q_t f(x) = \tilde{E}^x[f(\hat{Z}_t)]$ for x in A and $Q_t f(x) = 0$ for x outside A. From the definition of $\tilde{E}^x$ it follows that

$$Q_t f(x) = E^x[f(\hat{X}_t)\overline{G}(\hat{X}_t)]/\overline{G}(\hat{X}_0)] \quad \text{for } x \text{ in } A \text{ and equals to}$$

zero for x outside A.

For functions in $\mathcal{B}_0$ the infinitsimal generator

$$\mathcal{G}f = \lim_{t \to 0} \frac{Q_t f - f}{t}, \text{ where the limit is taken to be the uniform limit.}$$

We let $D(\mathcal{G}) = \{f \in \mathcal{B}_0 : \mathcal{G}f \text{ exists}\}$. Then, $D(\mathcal{G})$ is called the domain of the generator.

For any function $f \in \mathcal{B}_0$ we define $h: R_+ \times R_+ \to R$ by

$$h(x,y) = \overline{G}^{-1}(x)[f(x+y)\overline{G}(x+y) - f(x)\overline{G}(x)],$$

and note that $\sup_{x,y}|h(x,y)| \le 2\|f\|$. Define the operator $T: b\mathcal{B}(A) \to \mathcal{B}(A)$ by

$$Tf(x) = \int_{R_0} h(x,y)\mu(dy).$$

Note that T is not necessarily bounded, since μ is not necessarily finite. However, if $\overline{G}$ is differentiable on A and the failure rate is bounded, then T is bounded on the set of function $f \in \mathcal{B}_0$ such that $f' \in \mathcal{B}_0$. To see this observe that adding and subtracting $[f(x)\overline{G}(x+y)]/\overline{G}(x)$ to the definition of h and using Lagrange's Theorem we have that

$$h(x,y) = [y/\overline{G}(x)][-\overline{G}'(x + \theta_1 y)f(x) - f'(x + \theta_2 y)\overline{G}(x + y)]$$
$$\leq y\{[-\overline{G}'(x + \theta_1 y)/\overline{G}(x + \theta_1 y)]f(x) - f'(x + \theta_2 y)[\overline{G}(x + y)/\overline{G}(x)]\}.$$

It follows that for each $f \in B_0$ with $f' \in B_0$ we have that, for each $y \in R_+$,

$$\sup_x |h(x,y)| \leq y[\|f'\| + \|f\| r].$$
$$= y M,$$

where r is the supremum of the failure rate and $M = [\|f'\| + \|f\| r]$.

For such functions we therefore have that

$$\|Tf\| = \|\int_{R_0} h(x,y)\mu(dy)\|$$
$$\leq M\int_0^a y\mu(dy) + 2\|f\|\int_a^\infty \mu(dy)$$
$$< \infty.$$

The last inequality is true since for Lévy measure μ we have that $\int_{R_0} (y \wedge 1)\mu(dy) < \infty$ which is true if and only if $\int_0^1 y\mu(dy) < \infty$ and $\int_1^\infty \mu(dy) < \infty$. Thus, for $a > 1$, $\int_0^a y\mu(dy) = \int_0^1 y\mu(dy) + \int_1^a y\mu(dy)$ $< \int_0^1 y\mu(dy) + a\int_1^a \mu(dy) < \infty$; and $\int_a^\infty \mu(dy) < \infty$. This gives us the inequality.

The following theorem illustrates the fact that if $\overline{G}$ is differentiable with bounded continuous failure rate, on its support, then $\{f \in B_0 : f' \in b.c.B_0\} \subset D(G)$ and $Gf = Tf$.

3.3. THEOREM. Assume that $\overline{G}$ is differentiable with bounded almost everywhere continuous failure rate on its support. Then,

$$H \overset{\text{def}}{=} \{f \in bB_0 : f' \in b.c.B_0\} \subset D(G)$$

and

$$Gf = Tf.$$

PROOF. Consider for $f \in H$

$$\left\|\frac{Q_t f - f}{t} - Tf\right\| = \left\|\int_{R_0} h(x,y)\left[\frac{1}{t} Q_t(dy) - \mu(dy)\right]\right\| .$$

Let $\Omega_t(dy) = \frac{1}{t}\, yQ_t(dy)$. By an argument similar to the one used in proving Theorem 1, page 302 of Feller [8], it follows that $\Omega_{1/n}\{[0,a)\}$ is bounded for any $a \in R_+$ and $nQ_{1/n}\{[a^0,\infty)\} < \varepsilon$ for all n for some $a^0 \in R_+$. By Helly's Theorem there exists a sequence (n_k) such that as t^{-1} runs through it, the measures $\Omega_t(dy)$ converges weakly to some measure Ω over finite intervals. Therefore, to see that the limiting measure Ω equals $y\mu$ observe that for each $\lambda > 0$, the Laplace transform of $\hat{X}$ is of the form

$$E(e^{-\lambda \hat{X}_t}) = e^{-t\int_{R_0} (1 - e^{-\lambda y})\mu(dy)}$$

observe that for each h, $\left|e^{-\lambda y} \frac{(1 - e^{-hy})}{h}\right| \le \frac{1}{\lambda}(y \wedge 1)$,

the right-hand side of the inequality is integrable with respect to μ by properties of the Lévy measure. Therefore, by the LEBESGUE DOMINATED CONVERGENCE THEOREM it follows that

$$-\frac{d}{d\lambda}\int_{R_0} (1 - e^{-\lambda y})\mu(dy) = \int_{R_0} e^{-\lambda y} y\mu(dy),$$

which is the Laplace transform of $y\mu$. Upon differentiating each side of the Laplace transform formula above with respect to λ and dividing through by t and taking limits as t approaches zero we get

$$\lim_{t\to 0} \int_{R_0} e^{-\lambda y} \frac{1}{t} yQ_t(dy) = \int_{R_0} e^{-\lambda y} y\mu(dy)$$

Since $\frac{1}{t} yQ_t$ converges weakly over finite intervals and by the uniqueness of the Laplace transform the limiting measure is $\Omega = y\mu$.

Let $u(x,y) = \overline{G}^{-1}(x)[-\overline{G}'(x + \theta_1 y)f(x) - f'(x + \theta_2 y)\overline{G}(x + y)]$. Then, $\int_0^a h(x,y)\frac{1}{t} Q_t(dy) = \int_0^a u(x,y)y \frac{1}{t} Q_t(dy)$ the last term converges $\int_0^a u(x,y)y\mu(dy)$ at each continuity point a of $y\mu(dy)$. This convergence is uniform since u is continuous. Then, for any $f \in H$ we have

$$\left\|\frac{Q_t f - f}{f} - Tf\right\| = \left\|\int_{R_0} h(x,y)[\tfrac{1}{t} Q_t(dy) - \mu(dy)]\right\|.$$

[illegible]r any point of continuity [illegible] of [illegible] that exceeds [illegible] we have that the last term is less than or equal to $\int$ [illegible] $\frac{1}{t}$ [illegible] (dy) - [illegible]

$+ \int_a^x h(x,y)\frac{1}{t} Q_t(dy)$ [illegible] $+ \int_a$ [illegible] $h(x,y)$ [illegible] (dy) . The first term goes to zero as t^{-1} varies over (n_k). The second term is less than or equal to $2\|f\| \frac{1}{t} Q_t([a,\infty)) \leq 2\|f\| \varepsilon$ when t^{-1} varies over (n_k). The third term is less than or equal to $2\|f\|$ [illegible]$[a,\infty)$[illegible], this term can be made small enough by choosing a large enough. Therefore, we conclude that $\left\|\frac{Q_t f - f}{t} - Tf\right\| \to 0$ as $t \to 0$ through (n_k). The result then follows from Lemma 3, page 297 of Feller [8]. ||

The proof of the following proposition is obvious and hence is omitted.

3.4. <u>PROPOSITION</u>. Let $b = \inf_{\tau \in U} \psi_\tau$ and assume that $b > 0$. Then the following two problems are equivalent in the sense that they have the same solution:

(P_1) minimize ψ_τ over all $\tau \in U$

(P_2) maximize $\beta_\tau \overset{\text{def}}{=} b\tilde{E}^x(\tau) + \tilde{E}^x(c_1(Z_\tau))$ for all $\tau \in U$.

Finally, we state and prove the key theorem in this section.

3.5. <u>THEOREM</u>. Let $c \in \{f \in \beta_o : f' \in \text{b.c. } \beta_o\}$. Suppose that $b + Gc_1(x)$ crosses the x-axis at most once and if once then the crossing is from above. Assume that $\tilde{E}^x(\zeta) < \infty$ for all $x \in R_+$. Define

$$\alpha = \inf\{x : b + Gc_1(x) \leq 0\}.$$

Then

$$\tau^* = \inf\{t \geq 0 : Z_t \in [\alpha,\infty]\}$$

is the solution to (P_2) and hence to (P_1).

PROOF. By Remark 3.1, it suffices to prove the result for the stationary process $\hat{Z}$. Let $\hat{\tau}^* = \inf\{t: \hat{Z}_t \in [\alpha,\infty]\}$ and note that $\hat{\tau}$ does not exceed $\hat{\zeta}$. By assumptions, it follows that $\widetilde{E}^x(\hat{\tau})<\infty$. It is therefore clear that $\hat{\tau}$ is in $\hat{U}$. From Theorem 2.11.2 of Ito [9] it follows that for each $\hat{\tau} \in \hat{U}$,

$$\beta_{\hat{\tau}}(x) = \widetilde{E}^x \int_0^{\hat{\tau}} (b + Gc_1(\hat{Z}_s))dx + c_1(x)$$

for each $x \in R_+$. Since $\hat{\tau}^*$ belongs to $\hat{U}$, then for each $\hat{\tau} \in \hat{U}$ we have

that $\beta_{\hat{\tau}^*}(x) - \beta_{\hat{\tau}}(x) = \widetilde{E}^x[\int_0^{\hat{\tau}^*} (b + Gc_1(\hat{Z}_s))ds - \int_0^{\hat{\tau}} (b + Gc_1(\hat{Z}_s))ds]$.

The right-hand side equals $\widetilde{E}^x[\int_{\hat{\tau}}^{\hat{\tau}^*} (b + Gc_1(\hat{Z}_s))dsI_{(\hat{\tau}^* \geq \hat{\tau})}]$ $- \widetilde{E}^x[\int_{\hat{\tau}^*}^{\hat{\tau}} (b + Gc_1(\hat{Z}_s))dsI_{(\hat{\tau}^* < \hat{\tau})}]$. The first term is positive by definition of $\hat{\tau}^*$. The second term is also positive by definition of $\hat{\tau}^*$ and since $\hat{Z}$ is increasing. This finishes the proof.

4.6. EXAMPLES. (a) Assume that

$$\bar{G}(y) = \begin{cases} 1 & \text{if } y < y^o \\ 0 & \text{if } y \geq y^o \end{cases}$$

Then,

$$Gc_1(x) = \int_{R_0} (c_1(x+y) - c_1(x)) \, I_{[0,y^o-x)}(y)\mu(dy),$$

and the optimal replacement policy is a control policy if $c(x)$ is strictly convex.

(b) Assume that $\bar{G}(y) = e^{-y}$, $0 \leq y < y^o$, and zero otherwise. Thus, $A = (0,y^o)$ and we have that

$$Gc_1(x) = \int_A (e^{-y}c_1(x+y) - c_1(x))\mu(dy), \quad x \in A,$$

and zero otherwise.

and the optimal replacement policy is easily seen to be a control policy if $c(x)$ is convex strictly increasing. Since convex increasing functions approaches infinity as x approaches infinity, and $c(x)$ is bounded then A must be finite.

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